

# Hyperbolic Twistor Spaces

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*Dedicated to Professor Paulette Libermann*

**Abstract.** In contrast to the classical twistor spaces whose fibres are 2-spheres, we introduce twistor spaces over manifolds with almost quaternionic structures of the second kind in the sense of P. Libermann whose fibres are hyperbolic planes. We discuss two natural almost complex structures on such a twistor space and their holomorphic functions.

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## 1. Introduction

In this paper we introduce hyperbolic twistor spaces which are bundles over manifolds with almost quaternionic structure of the second kind in the sense of P. Libermann and whose fibres are hyperbolic planes. These spaces admit two natural almost complex structures defined as in the classical twistor space theory. Our main purpose is to study their differential-geometric properties as well as the existence of holomorphic functions (part of the results have been announced in the first author's lecture [4]). In Section 2 we define hyperbolic twistor spaces and their almost complex structures when the base manifolds are of dimension  $\geq 8$ , developing the theory for paraquaternionic Kähler manifolds. In Section 3 we give the corresponding treatment for base manifolds which are 4-dimensional with a metric of signature  $(++--)$ . Finally in Section 4 we treat the question of existence of holomorphic functions on hyperbolic twistor spaces and show that, in contrast to the classical case, there can be an abundance of (global) holomorphic functions on a hyperbolic twistor space.

## 2. Hyperbolic twistor spaces

We begin with the following simple observation. In [15] P. Libermann introduced the notion of an *almost quaternionic structure of the second kind* (*presque*

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*quaternioniennes de deuxième espèce*) on a smooth manifold  $M$ . This consists of an almost complex structure  $J_1$  and an almost product structure,  $J_2$  such that  $J_1 J_2 + J_2 J_1 = 0$ . Setting  $J_3 = J_1 J_2$  one has a second almost product structure which also anti-commutes with  $J_1$  and  $J_2$ . Now on a manifold  $M$  with such a structure, set

$$j = y_1 J_1 + y_2 J_2 + y_3 J_3.$$

Then  $j$  is an almost complex structure on  $M$  if and only if

$$-y_1^2 + y_2^2 + y_3^2 = -1$$

which suggests considering a *hyperbolic twistor space*  $\pi : \mathcal{Z} \longrightarrow M$  with fibre this hyperboloid. Recall that the classical twistor space over a quaternionic Kähler manifold is a bundle over the manifold with the fibre being a 2-sphere (Salamon [18]).

An *almost paraquaternionic structure* on a smooth manifold  $M$  is defined to be a rank 3-subbundle  $E$  of the endomorphisms bundle  $\text{End}(TM)$  which locally is spanned by a triple  $\{J_1, J_2, J_3\}$  which is an almost quaternionic structure of the second kind in the sense of P. Libermann.

There are a number of examples of almost paraquaternionic structures including the paraquaternionic projective space as described by Blažić [5]. Under certain holonomy assumptions almost paraquaternionic structures become paraquaternionic Kähler (see e.g. Garcia-Rio, Matsushita and Vazquez-Lorenzo [9]). Even more strongly one has the notion of a neutral hyperkähler structure (see Section 4) and Kamada [14] has observed that the only compact four-manifolds admitting such a structure are complex tori and primary Kodaira surfaces. We remark that the neutral hyperkähler four-manifolds are Ricci flat and self-dual ([14]).

The tangent bundle of a differentiable manifold also carries an almost paraquaternionic structure as studied by S. Ianus and C. Udriste [11] [12]; this includes examples where the dimension of the manifold carrying the structure is not necessarily  $4n$ . However the most natural setting for this kind of structure is on a manifold  $M$  of dimension  $4n$  with a neutral metric  $g$ , i.e. a pseudo-Riemannian metric of signature  $(2n, 2n)$ . One reason for this is that such a metric may be given with respect to which  $J_1$  acts as an isometry on tangent spaces and  $J_2, J_3$  act as anti-isometries; the effect of this is that we may define three fundamental 2-forms  $\Omega_a$ ,  $a = 1, 2, 3$ , by  $\Omega_a(X, Y) = g(X, J_a Y)$ . If a neutral metric  $g$  has this property we shall say that it is *adapted* to the almost paraquaternionic structure  $E$ ; we shall also say that  $J_1, J_2, J_3$  are *compatible* with  $g$ . Riemannian metrics can be chosen such that  $g(J_a X, J_a Y) = g(X, Y)$ , but then  $\Omega_2$  and  $\Omega_3$  are symmetric tensor fields instead of 2-forms.

The neutral metric  $g$  induces a metric on the fibres of  $E$  by  $\frac{1}{4n} \text{tr} A^t B$  where  $A$  and  $B$  are endomorphisms of  $T_p M$  and  $A^t$  is the adjoint of  $A$  with respect to  $g$ . This metric on the fibre is of signature  $(+ - -)$ , the norm of  $J_1$  being  $+1$  and the norms of  $J_2$  and  $J_3$  being  $-1$ . The twistor space  $\mathcal{Z}$  of an almost paraquaternionic structure  $E$  with an adapted neutral metric  $g$  is the unit sphere subbundle of  $E$  (with respect to the induced metric).

Alternatively one may take the Lorentz metric  $\langle, \rangle$  on the fibres of  $E$  such that  $\langle J_1, J_1 \rangle = -1$ ,  $\langle J_2, J_2 \rangle = +1$ ,  $\langle J_3, J_3 \rangle = +1$ . This metric is of signature  $(-++)$  and has the advantage of inducing immediately a Riemannian metric of constant curvature  $-1$  on the hyperbolic planes defined by  $-y_1^2 + y_2^2 + y_3^2 = -1$ , in each fibre. We adopt this metric for its geometric attractiveness but keep its negative in mind.

We will also use the following notation. For the metric  $\langle, \rangle$  on the fibres of  $E$  we set  $\epsilon_1 = -1$  and  $\epsilon_2 = \epsilon_3 = +1$ . For the neutral metric  $g$  of on the base, we set  $\epsilon_i = \pm 1$  according to the signature  $(+\cdots + - \cdots -)$ . Further, denoting also by  $\pi$  the projection of  $E$  onto  $M$ , if  $x_i$  are local coordinates on  $M$ , set  $q_i = x_i \circ \pi$ . We will identify the tangent space of  $E$  at a point  $x \in E$  with the fibre  $E_{\pi(x)}$  through that point. For a section  $s$  of  $E$  we denote its vertical lift to  $E$  as a vector field by  $s^v$  (so  $s^v = s \circ \pi$ ) and frequently utilize the natural identifications of  $J_a^v$  with  $J_a$  itself and with  $\frac{\partial}{\partial y_a}$  in terms of the fibre coordinates  $y_1, y_2, y_3$ .

An almost paraquaternionic manifold  $M$  of dimension  $4n$  and neutral metric  $g$  is said to be *paraquaternionic Kähler* if the bundle  $E$  is parallel with respect to the Levi-Civita connection of  $g$ .

As with the theory of twistor spaces over quaternionic Kähler manifolds, the theory of hyperbolic twistor spaces over paraquaternionic Kähler manifolds develops nicely by virtue of the fact that the covariant derivatives of sections of  $E$  are again sections of  $E$ . To give this development we first need the natural machinery of horizontal lifts.

Let  $D$  denote the Levi-Civita connection of the neutral metric on  $M$ . Then the horizontal lift  $X^h$  of a vector field  $X$  to the bundle  $\pi : E \rightarrow M$  is given by

$$X^h = \sum_i X^i \frac{\partial}{\partial q^i} - \sum_{a,b=1}^3 \epsilon_b y_a (\langle D_X J_a, J_b \rangle \circ \pi) \frac{\partial}{\partial y_b}. \quad (2.1)$$

It is straightforward to obtain the following at a point  $\sigma \in Z \subset E$

$$[X^h, Y^h]_\sigma = [X, Y]_\sigma^h - R(X, Y)\sigma$$

(we adopt here the following definition of the curvature tensor:  $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ ). For a section  $s$  of  $E$ ,

$$[X^h, s^v] = (D_X s)^v.$$

Define a metric on  $E$  by  $h_t = \pi^* g + t \langle, \rangle$ ,  $t \neq 0$ , and for simplicity denote its Levi-Civita connection by  $\bar{\nabla}$  instead of  $\bar{\nabla}^t$ , but note the  $t$  in the formulas below. Then

$$(\bar{\nabla}_{X^h} Y^h)_\sigma = (D_X Y)_\sigma^h - \frac{1}{2} R(X, Y)\sigma, \quad (2.2)$$

$$(\bar{\nabla}_{X^h} s^v)_\sigma = \frac{t}{2} (\hat{R}_{\sigma s} X)^h + (D_X s)_\sigma^v \quad (2.3)$$

where  $g(\hat{R}_{\sigma s} X, Y) = \langle R(X, Y)\sigma, s^v \rangle$  and

$$(\bar{\nabla}_{s^v} X^h)_\sigma = \frac{t}{2} (\hat{R}_{\sigma s} X)^h, \quad \bar{\nabla}_{k^v} s^v = 0. \quad (2.4)$$

for sections  $k$  and  $s$  of  $E$ .

**Lemma 1** *The Weingarten map  $A_t$  of the hyperbolic twistor space  $Z$ , as a hypersurface in the bundle space  $E$ , annihilates horizontal vectors and acts on vertical vectors by  $A_t V = \frac{\sqrt{|t|}}{t} V$ .*

**Proof:** The position vector  $\sigma$  gives rise to a normal  $\nu_t = \frac{\sigma}{\sqrt{|t|}}$  to the hyperboloid in the fibres of  $E$  and the Weingarten map is given by  $A_t X = \frac{|t|}{t} \bar{\nabla}_X \nu_t$ . Using the summation convention for repeated indices  $a, b, c = 1, 2, 3$ , we have first for a horizontal lift (equation (2.1)) to a point  $\sigma$

$$\begin{aligned} \bar{\nabla}_{X^h} \nu_t &= \frac{1}{\sqrt{|t|}} \bar{\nabla}_{X^h y_c} \frac{\partial}{\partial y_c} \\ &= \frac{1}{\sqrt{|t|}} \left( -\epsilon_c y_a (\langle D_X J_a, J_c \rangle \circ \pi) \frac{\partial}{\partial y_c} \right) + \frac{1}{\sqrt{|t|}} y_c \left( \frac{t}{2} (\hat{R}_{\sigma J_c} X)^h + (D_X J_c)_\sigma \right) \\ &= \frac{t}{2\sqrt{|t|}} (\hat{R}_{\sigma\sigma} X)^h = 0. \end{aligned}$$

Similarly for a vertical tangent vector  $V$

$$\bar{\nabla}_V \nu_t = \frac{1}{\sqrt{|t|}} \bar{\nabla}_V y_c \frac{\partial}{\partial y_c} = \frac{1}{\sqrt{|t|}} (V y_c) \frac{\partial}{\partial y_c} = \frac{V^c}{\sqrt{|t|}} \frac{\partial}{\partial y_c} = \frac{1}{\sqrt{|t|}} V.$$

Thus  $A_t V = \frac{\sqrt{|t|}}{t} V$  for a vertical tangent vector  $V$  and  $A_t X = 0$  for a horizontal tangent vector  $X$ .  $\blacksquare$

We now define two almost complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$  on the hyperbolic twistor space  $\mathcal{Z}$  as follows. Acting on horizontal vectors these are the same and given by  $\mathcal{J}_1 X_\sigma^h = \mathcal{J}_2 X_\sigma^h = (jX)_\sigma^h$  where as before  $j = \sum y_a J_a$  is the point  $\sigma$  considered as an endomorphism of  $TM$ . For a vertical vector  $V = V^1 \frac{\partial}{\partial y_1} + V^2 \frac{\partial}{\partial y_2} + V^3 \frac{\partial}{\partial y_3}$  tangent to  $\mathcal{Z}$ , i.e.  $\langle \sigma, V \rangle = 0$ , let

$$\mathcal{J}_1 V = (y_3 V^2 - y_2 V^3) \frac{\partial}{\partial y_1} + (y_3 V^1 - y_1 V^3) \frac{\partial}{\partial y_2} + (y_1 V^2 - y_2 V^1) \frac{\partial}{\partial y_3} \quad (2.5)$$

and let  $\mathcal{J}_2 V$  be the negative of this expression. At each point  $p \in M$ , the local endomorphisms  $\{J_1, J_2, J_3\}$  form an orthonormal basis of the fibre  $E_p$  of  $E$  over  $p$  and define the same orientation on it. Denote by  $\times$  the vector product on the 3-dimensional vector space  $E_p$  determined by this orientation and the metric of  $E$  (in other words determined by the paraquaternionic algebra). Then  $\mathcal{J}_k V = (-1)^{k-1} \sigma \times V$ ,  $k = 1, 2$ , for any  $\sigma \in \mathcal{Z}$

Define a pseudo-Riemannian metric on  $\mathcal{Z}$  by  $h_t = \pi^* g + t \langle \cdot, \cdot \rangle_v$ ,  $t \neq 0$ ,  $\langle \cdot, \cdot \rangle_v$  being the restriction of  $\langle \cdot, \cdot \rangle$  to the fibres (hyperbolic planes) of  $\mathcal{Z}$  and denote the

Levi-Civita connection of  $h_t$  by  $\nabla$  for simplicity. It is easy to check that this metric is Hermitian with respect to both  $\mathcal{J}_1$  and  $\mathcal{J}_2$ .

We now review paraquaternionic Kähler geometry; the development of theory of paraquaternionic Kähler structures was carried out by Garcia-Rio, Matsushita and Vazquez-Lorenzo [9] when the dimension of the base manifold is  $4n \geq 8$ . As with the theory of quaternionic Kähler manifolds, dimension 4 is special. At the beginning however we will retain the 4-dimensional case and point out where the differences occur. The parallel to the present development in the quaternionic Kähler case can be found in Ishihara [13].

Let  $\{J_1, J_2, J_3\}$  be a local almost quaternionic structure of the second kind which spans the bundle  $E$ . Since  $E$  is parallel with respect to  $D$ , there exist local 1-forms  $\alpha, \beta$  and  $\gamma$  such that

$$\begin{aligned} D_X J_1 &= -\gamma(X)J_2 - \beta(X)J_3, \\ D_X J_2 &= -\gamma(X)J_1 - \alpha(X)J_3, \\ D_X J_3 &= -\beta(X)J_1 + \alpha(X)J_2. \end{aligned} \tag{2.6}$$

From the group theoretic point of view, this structure corresponds to the linear holonomy group being a subgroup of  $Sp(n, \mathbb{R}) \cdot Sp(1, \mathbb{R})$ , just as a quaternionic Kähler structure corresponds to the linear holonomy group being a subgroup of  $Sp(n) \cdot Sp(1)$ . For  $n = 1$  this is not a restriction.

Setting

$$A = 2(d\alpha - \beta \wedge \gamma), \quad B = 2(d\beta - \alpha \wedge \gamma), \quad C = 2(d\gamma + \alpha \wedge \beta)$$

one can easily obtain the following central relation of the action of the curvature tensor:

$$\begin{aligned} R(X, Y)J_1 &= -C(X, Y)J_2 - B(X, Y)J_3, \\ R(X, Y)J_2 &= -C(X, Y)J_1 - A(X, Y)J_3, \\ R(X, Y)J_3 &= -B(X, Y)J_1 + A(X, Y)J_2. \end{aligned} \tag{2.7}$$

Moreover a paraquaternionic Kähler manifold of dimension  $\geq 8$  is Einstein and  $A, B, C$  satisfy

$$A(X, Y) = -\frac{\tau g(X, J_1 Y)}{4n(n+2)}, \quad B(X, Y) = -\frac{\tau g(X, J_2 Y)}{4n(n+2)}, \quad C(X, Y) = \frac{\tau g(X, J_3 Y)}{4n(n+2)} \tag{2.8}$$

where  $\tau$  is the scalar curvature of the metric  $g$ .

We now give our first result.

**Theorem 1** *On the hyperbolic twistor space of a paraquaternionic Kähler manifold of dimension  $4n \geq 8$  we have the following:*

- (i) *The almost complex structure  $\mathcal{J}_1$  is integrable and the Hermitian structure  $(\mathcal{J}_1, h_t)$  is semi-Kähler. It is indefinite Kähler if and only if  $t\tau = -4n(n+2)$ .*

(ii) The almost complex structure  $\mathcal{J}_2$  is never integrable but the almost Hermitian structure  $(\mathcal{J}_2, h_t)$  is semi-Kähler. It is indefinite almost Kähler if and only if  $t\tau = 4n(n+2)$  and indefinite nearly Kähler if and only if  $t\tau = -2n(n+2)$ .

**Proof:** The major effort of the proof is to compute the covariant derivatives of  $\mathcal{J}_i$ ,  $i = 1, 2$ . To begin, by Lemma 1 and (2.2)

$$\begin{aligned} (\nabla_{X^h} \mathcal{J}_i) Y^h \big|_\sigma &= \bar{\nabla}_{X^h} \mathcal{J}_i Y^h - \mathcal{J}_i \bar{\nabla}_{X^h} Y^h \\ &= \bar{\nabla}_{X^h} (y_1 J_1 Y + y_2 J_2 Y + y_3 J_3 Y)^h - \mathcal{J}_i (D_X Y)^h + \frac{1}{2} \mathcal{J}_i R(X, Y) \sigma \end{aligned}$$

Expanding further and using (2.1) and (2.7),

$$\begin{aligned} (\nabla_{X^h} \mathcal{J}_i) Y^h \big|_\sigma &= -\frac{1}{2} y_1 R(X, J_1 Y) \sigma - \frac{1}{2} y_2 R(X, J_2 Y) \sigma - \frac{1}{2} y_3 R(X, J_3 Y) \sigma \\ &+ \frac{1}{2} \mathcal{J}_i \left[ y_1 \left( -C(X, Y) \frac{\partial}{\partial y_2} - B(X, Y) \frac{\partial}{\partial y_3} \right) + y_2 \left( -C(X, Y) \frac{\partial}{\partial y_1} - A(X, Y) \frac{\partial}{\partial y_3} \right) \right. \\ &\quad \left. + y_3 \left( -B(X, Y) \frac{\partial}{\partial y_1} + A(X, Y) \frac{\partial}{\partial y_2} \right) \right]. \end{aligned}$$

Note that  $y_1 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_1}$ , etc. are tangent to the fibres. Applying  $\mathcal{J}_i$  and expanding the curvature terms by (2.7) we have

$$(\nabla_{X^h} \mathcal{J}_1) Y^h \big|_\sigma = 0 \tag{2.9}$$

and

$$\begin{aligned} (\nabla_{X^h} \mathcal{J}_2) Y^h \big|_\sigma &= - \left[ ((y_2^2 + y_3^2) A(X, Y) + y_1 y_2 B(X, Y) - y_1 y_3 C(X, Y)) \frac{\partial}{\partial y_1} \right. \\ &\quad + ((y_1 y_2 A(X, Y) + (y_1^2 - y_3^2) B(X, Y) - y_2 y_3 C(X, Y)) \frac{\partial}{\partial y_2} \\ &\quad \left. + ((y_1 y_3 A(X, Y) + y_2 y_3 B(X, Y) + (y_2^2 - y_1^2) C(X, Y)) \frac{\partial}{\partial y_3} \right] \\ &= \frac{\tau}{4n(n+2)} \left[ (-g(X, J_1 Y) + y_1 g(X, jY)) \frac{\partial}{\partial y_1} \right. \\ &\quad \left. + (g(X, J_2 Y) + y_2 g(X, jY)) \frac{\partial}{\partial y_2} + (g(X, J_3 Y) + y_3 g(X, jY)) \frac{\partial}{\partial y_3} \right] \end{aligned}$$

using (2.8) and  $-y_1^2 + y_2^2 + y_3^2 = -1$ . Taking the inner product with a vertical tangent vector  $V$  we have

$$h_t((\nabla_{X^h} \mathcal{J}_2) Y^h, V) = \frac{t\tau}{4n(n+2)} [V^1 g(X, J_1 Y) + V^2 g(X, J_2 Y) + V^3 g(X, J_3 Y)]. \tag{2.10}$$

For  $(\nabla_{X^h} \mathcal{J}_i)V$ , its horizontal part may be found immediately from the above and to show that its vertical part vanishes we show that  $(\nabla_{X^h} \mathcal{J}_i)V$  is horizontal. To do this effectively recall that  $\mathcal{J}_1$  can be given by equation (2.5) and we may regard this formula as extended to  $E$ , i.e.  $\mathcal{J}_1 V$  is given by this formula for  $V$  tangent to  $E$ , even though one no longer has  $\mathcal{J}_1^2 = -I$ . Then

$$\begin{aligned}
(\nabla_{X^h} \mathcal{J}_1) \frac{\partial}{\partial y_1} &= \bar{\nabla}_{X^h} \left( y_3 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_3} \right) - \mathcal{J}_1 \left( \frac{t}{2} (\hat{R}_{\sigma J_1} X)^h - \gamma(X) \frac{\partial}{\partial y_2} - \beta(X) \frac{\partial}{\partial y_3} \right) \\
&= (y_1 \beta(X) + y_2 \alpha(X)) \frac{\partial}{\partial y_2} + y_3 \left( \frac{t}{2} (\hat{R}_{\sigma J_2} X)^h - \gamma(X) \frac{\partial}{\partial y_1} - \alpha(X) \frac{\partial}{\partial y_3} \right) \\
&+ (-y_1 \gamma(X) + y_3 \alpha(X)) \frac{\partial}{\partial y_3} - y_2 \left( \frac{t}{2} (\hat{R}_{\sigma J_3} X)^h - \beta(X) \frac{\partial}{\partial y_1} + \alpha(X) \frac{\partial}{\partial y_2} \right) \\
&- \frac{t}{2} (j \hat{R}_{\sigma J_1} X)^h + \gamma(X) \left( y_3 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_3} \right) + \beta(X) \left( -y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \right) \\
&= \frac{t}{2} y_3 (\hat{R}_{\sigma J_2} X)^h - \frac{t}{2} y_2 (\hat{R}_{\sigma J_3} X)^h - \frac{t}{2} (j \hat{R}_{\sigma J_1} X)^h
\end{aligned}$$

which is horizontal. The proof for  $\frac{\partial}{\partial y_2}$  and  $\frac{\partial}{\partial y_3}$  and for  $\mathcal{J}_2$  is similar.

Similarly treating  $(\nabla_V \mathcal{J}_i)X^h$  we find that

$$\begin{aligned}
&h_t((\nabla_V \mathcal{J}_i)X^h, Y^h) \\
&= \frac{4n(n+2) + t\tau}{4n(n+2)} [V^1 g(J_1 X, Y) + V^2 g(J_2 X, Y) + V^3 g(J_3 X, Y)]. \quad (2.11)
\end{aligned}$$

Finally for vertical tangent vectors  $V$  and  $W$

$$(\nabla_V \mathcal{J}_i)W = \bar{\nabla}_V \mathcal{J}_i W - t \langle A_t V, \mathcal{J}_i W \rangle \nu_t - \mathcal{J}_i \bar{\nabla}_V W$$

noting that the extension of  $\mathcal{J}_i$  to  $E$  annihilates  $\nu_t$ . Treating the terms separately for  $\mathcal{J}_1$

$$\begin{aligned}
\bar{\nabla}_V \mathcal{J}_1 W &= (V^3 W^2 + y_3 V W^2 - V^2 W^3 - y_2 V W^3) \frac{\partial}{\partial y_1} \\
&+ (V^3 W^1 + y_3 V W^1 - V^1 W^3 - y_1 V W^3) \frac{\partial}{\partial y_2} \\
&+ (V^1 W^2 + y_1 V W^2 - V^2 W^1 - y_2 V W^1) \frac{\partial}{\partial y_3}, \\
\mathcal{J}_1 \bar{\nabla}_V W &= (V W^1) \left( y_3 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_3} \right) + (V W^2) \left( y_3 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_3} \right) \\
&+ (V W^3) \left( -y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \right)
\end{aligned}$$

and using  $\langle \sigma, V \rangle = \langle \sigma, W \rangle = 0$  and  $-y_1^2 + y_2^2 + y_3^2 = -1$

$$y_1 \langle V, \mathcal{J}_1 W \rangle = V^3 W^2 - V^2 W^3,$$

$$\begin{aligned} y_2 \langle V, \mathcal{J}_1 W \rangle &= V^3 W^1 - V^1 W^3, \\ y_3 \langle V, \mathcal{J}_1 W \rangle &= V^1 W^2 - V^2 W^1. \end{aligned}$$

Combining these we have  $(\nabla_V \mathcal{J}_1)W = 0$  and similarly  $(\nabla_V \mathcal{J}_2)W = 0$ .

Using these computations we can now easily complete the proof of Theorem 1. That the almost Hermitian structure  $(\mathcal{J}_1, h)$  is Kähler if and only if  $t\tau = -4n(n+2)$  follows immediately from the above relations, especially equations (2.9) and (2.11). To show the integrability of  $\mathcal{J}_1$  first recall that the Nijenhuis tensor  $N_i$  of  $\mathcal{J}_i$

$$N_i(X, Y) = -[X, Y] + [\mathcal{J}_i X, \mathcal{J}_i Y] - \mathcal{J}_i[\mathcal{J}_i X, Y] - \mathcal{J}_i[X, \mathcal{J}_i Y]$$

may be written in terms of the connection  $\nabla$  as

$$N_i(X, Y) = \mathcal{J}_i(\nabla_Y \mathcal{J}_i)X - (\nabla_{\mathcal{J}_i Y} \mathcal{J}_i)X - \mathcal{J}_i(\nabla_X \mathcal{J}_i)Y + (\nabla_{\mathcal{J}_i X} \mathcal{J}_i)Y. \quad (2.12)$$

The cases  $N_1(X^h, Y^h) = 0$  and  $N_1(V, W) = 0$  are immediate. For  $N_1(V, X^h)$ , observe that the first two terms of the expansion (2.12) vanish while the remaining two are horizontal. Thus it is enough to compute

$$h_t(N_1(V, X^h), Y^h) = h_t((\nabla_V \mathcal{J}_1)X^h, (jY)^h) + h_t((\nabla_{\mathcal{J}_1 V} \mathcal{J}_1)X^h, Y^h);$$

upon expansion using (2.11) the two terms will cancel.

For the almost Hermitian structure  $(\mathcal{J}_2, h)$ , to see that it is almost Kähler if and only if  $t\tau = 4n(n+2)$ , the key case to consider is

$$\begin{aligned} &h_t((\nabla_{X^h} \mathcal{J}_2)Y^h, V) + h_t((\nabla_V \mathcal{J}_2)X^h, Y^h) + h_t((\nabla_{Y^h} \mathcal{J}_2)V, X^h) \\ &= \frac{t\tau - 4n(n+2)}{4n(n+2)} [V^1 g(X, J_1 Y) + V^2 g(X, J_2 Y) + V^3 g(X, J_3 Y)]. \end{aligned}$$

To see that  $(\mathcal{J}_2, h)$  is nearly Kähler if and only if  $t\tau = -2n(n+2)$ , note that  $h_t((\nabla_{X^h} \mathcal{J}_2)Y^h + (\nabla_{Y^h} \mathcal{J}_2)X^h, V) = 0$  by the skew-symmetry in equation (2.10) and by equations (2.10) and (2.11)

$$\begin{aligned} &h_t((\nabla_{X^h} \mathcal{J}_2)V + (\nabla_V \mathcal{J}_2)X^h, Y^h) \\ &= \frac{4n(n+2) + 2t\tau}{4n(n+2)} [V^1 g(J_1 X, Y) + V^2 g(J_2 X, Y) + V^3 g(J_3 X, Y)]. \end{aligned}$$

To show the non-integrability of  $\mathcal{J}_2$ , we compute  $h_t(N_2(V, X^h), Y^h)$  at the point  $(1, 0, 0)$  with  $V = \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}$ . The first term in the expansion (2.12) yields

$$-h_t((\nabla_{X^h} \mathcal{J}_2)V, (jY)^h) = \frac{t\tau}{4n(n+2)} [-g(X, J_3 Y) + g(X, J_2 Y)].$$

Proceeding in this way with the other terms we get

$$h_t(N_2(V, X^h), Y^h) = -2[g(X, J_2 Y) - g(X, J_3 Y)]$$



which is not identically zero, e.g take  $X = J_2 Y$ .

Finally, note that, by the above computations,  $(\nabla_{X^h} \mathcal{J}_i) X^h = 0$  for any  $X \in TM$  and  $(\nabla_V \mathcal{J}_i) V = 0$  for any vertical vector  $V$ ,  $i = 1, 2$ . This implies that  $(\mathcal{J}_i, h_t)$  has co-closed fundamental 2-form, i.e. it is semi-Kähler. ■

**Remark.** The values of the scalar curvature appearing in Theorem 1 for  $t = 1$  are the negatives of what one has in the usual twistor space over a quaternionic Kähler manifold of dimension  $\geq 8$ , see e.g. [1]. This sign change is due to our choice of metric on the fibres of  $E$ . If we take  $<, >$  as the  $(+ - -)$  metric we would have the other values, but the fibres of  $\mathcal{Z}$  would then have a negative definite metric. In the classical case the almost complex structure  $\mathcal{J}_1$  was introduced and shown to be integrable by S. Salamon [18] and independently by L. Bérard Bergery (unpublished but see e.g. Besse [3]).

### 3. The 4-dimensional case

Let  $M$  be an oriented 4-dimensional manifold with a neutral metric  $g$  and  $\mathbf{e}_1, \dots, \mathbf{e}_4$  a local orthonormal frame with  $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$  giving the orientation. The metric  $g$  induces a metric on bundle of bivectors,  $\bigwedge^2 TM$ , by

$$g(\mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_k \wedge \mathbf{e}_l) = \frac{1}{2} \begin{vmatrix} \varepsilon_i \delta_{ik} & \varepsilon_i \delta_{il} \\ \varepsilon_j \delta_{jk} & \varepsilon_j \delta_{jl} \end{vmatrix}, \quad \varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = -1.$$

The Hodge star operator of the neutral metric acting on  $\bigwedge^2 TM$  is given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Let  $\bigwedge^-$  and  $\bigwedge^+$  denote the subbundles of  $\bigwedge^2 TM$  determined by the corresponding eigenvalues of the Hodge star operator. The metrics induced on  $\bigwedge^-$  and  $\bigwedge^+$  have signature  $(+ - -)$ .

Setting

$$\begin{aligned} s_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, & \bar{s}_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \\ s_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4, & \bar{s}_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, \\ s_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3, & \bar{s}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3, \end{aligned}$$

$\{s_1, s_2, s_3\}$  and  $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$  are local oriented orthonormal frames for  $\bigwedge^-$  and  $\bigwedge^+$  respectively.

Reversing the orientation of  $M$  just interchanges the roles of  $\bigwedge^-$  and  $\bigwedge^+$ , and we shall concentrate only on the bundle  $\bigwedge^-$ .

Further we shall often identify  $\bigwedge^2 TM$  with the bundle of skew-symmetric endomorphisms of  $TM$  by the correspondence that assigns to each  $\sigma \in \bigwedge^2 TM$  the endomorphism  $K_\sigma$  on  $T_p M$ ,  $p = \pi(\sigma)$ , defined by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y); X, Y \in T_p M. \quad (3.1)$$

Thus  $s_1, s_2, s_3$  are identified with the endomorphisms representing in the frame  $\mathbf{e}_1, \dots, \mathbf{e}_4$  by the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Hence the bundle  $E = \bigwedge^-$  defines an almost paraquaternionic structure on  $M$ , the local endomorphisms  $\{J_1, J_2, J_3\}$  spanning  $E$  being  $J_1 = K_{s_1}, J_2 = K_{s_2}, J_3 = K_{s_3}$ . Moreover, the Levi-Civita connection of  $M$  preserves the bundle  $\bigwedge^-$ . So, as we have already mentioned, the existence of a paraquaternionic Kähler structure does not impose any restriction on the oriented Riemannian four-manifolds (the four-dimensional analog of paraquaternionic Kähler manifolds are the Einstein self-dual manifolds).

Now, in accordance with Section 2, the hyperbolic twistor space  $\mathcal{Z}$  of  $M$  is defined to be the unit sphere bundle in  $\bigwedge^-$ . It can be identified via (3.1) with the space of all complex structures on the tangent spaces of  $M$  compatible with its metric and orientation. We keep the notations  $\mathcal{J}_1$  and  $\mathcal{J}_2$  for the natural almost complex structures on  $\mathcal{Z}$  noting that in the Riemannian case they have been introduced and studied by Atiyah-Hitchin-Singer [2] and, respectively, Eells-Salamon [8].

Let  $\mathcal{R} : \bigwedge^2 TM \longrightarrow \bigwedge^2 TM$  be the curvature operator of  $(M, g)$ . It is related to the curvature tensor  $R$  by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = -g(R(X, Y)Z, T); X, Y, Z, T \in TM.$$

It is not hard to check that, for any  $a \in \bigwedge^2 TM$  and  $b, c \in \bigwedge^-$ , we have

$$g(R(a)b, c) = -g(\mathcal{R}(b \times c), a) \quad (3.2)$$

where  $R$  on the left-hand side stands for the curvature of the connection on the bundle  $\bigwedge^2 TM$  induced by the Levi-Civita connection of  $M$ .

Let us also note that if  $V \in \mathcal{V}_\sigma$  and  $X, Y \in T_{\pi(\sigma)}M$ , then

$$g(\sigma \times V, X \wedge Y) = -g(V, X \wedge K_\sigma Y). \quad (3.3)$$

The curvature operator  $\mathcal{R} : \bigwedge^2 TM \longrightarrow \bigwedge^2 TM$  admits an  $SO(2, 2)$ -irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6}I + \mathcal{B} + \mathcal{W}^+ + \mathcal{W}^-$$

similar to that in the 4-dimensional Riemannian case. Here  $\mathcal{B}$  represents the traceless Ricci tensor,  $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$  corresponds to the Weyl conformal tensor, and  $\mathcal{W}^\pm = \frac{1}{2}(\mathcal{W} \pm *\mathcal{W})$ . The metric  $g$  is said to be *self-dual* if  $\mathcal{W}^- = 0$ .

The metric  $g$  on the bundle  $\pi : \bigwedge^2 TM \longrightarrow M$  induced by the metric of  $M$  is negative definite on the fibres of  $\mathcal{Z}$  and, as in Section 2, we adopt the

metric  $\langle, \rangle = -g$  on  $\bigwedge^2 TM$ . Setting  $h_t = \pi^*g + t\langle, \rangle$  for any real  $t \neq 0$  we get a 1-parameter family of pseudo-Riemannian metrics on  $\mathcal{Z}$  compatible with the almost complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$

Again for simplicity we denote by  $\nabla$  the Levi-Civita connection of  $(\mathcal{Z}, h_t)$  and let  $D$  be the Levi-Civita connection of  $(M, g)$ .

Let  $X, Y$  be vector fields on  $M$  and  $V$  a vertical vector field on  $\mathcal{Z}$ . Then, for any point  $\sigma \in \mathcal{Z}$ ,

$$(\nabla_{X^h} Y^h)_\sigma = (D_X Y)_\sigma^h - \frac{1}{2} R(X, Y)_\sigma, \quad (3.4)$$

$$(\nabla_V X^h)_\sigma = \mathcal{H}(\nabla_{X^h} V)_\sigma = -\frac{t}{2} (R(\sigma \times V) X)_\sigma^h. \quad (3.5)$$

where  $\mathcal{H}$  means "the horizontal component".

Indeed, the first identity is a consequence of the standard formula for the Levi-Civita connection and the fact that  $[X^h, Y^h]_\sigma = [X, Y]_\sigma^h - R(X, Y)_\sigma$ ,  $\sigma \in \mathcal{Z}$ . To see (3.5), let us note that  $\nabla_V X^h$  is a horizontal vector field since the fibres of  $\mathcal{Z}$  are totally geodesic submanifolds. On the other hand,  $[V, X^h]$  is a vertical vector field, hence  $\nabla_V X^h = \mathcal{H}\nabla_{X^h} V$ . Then, by (3.2), we have

$$\begin{aligned} h_t(\nabla_V X^h, Y^h) &= h_t(\nabla_{X^h} V, Y^h) = -h_t(V, \nabla_{X^h} Y^h) = -\frac{t}{2} g(R(X, Y)_\sigma, V) = \\ &= \frac{t}{2} g(\mathcal{R}(\sigma \times V), X \wedge Y) = -\frac{t}{2} h_t((R(\sigma \times V) X)_\sigma^h, Y_\sigma^h) \end{aligned}$$

and we get the second identity in (3.5).

We are now going to compute the covariant derivative  $\nabla \mathcal{J}_k$  of the almost complex structure  $\mathcal{J}_k$  on the twistor space  $\mathcal{Z}$ ,  $k = 1, 2$ . The computation is similar to that in [10, 16] and we present it here for completeness.

Let  $\Omega_{k,t}(A, B) = h_t(A, \mathcal{J}_k B)$  be the fundamental 2-form of the almost Hermitian structure  $(\mathcal{J}_k, h_t)$ .

**Lemma 2** *Let  $\sigma \in \mathcal{Z}$ ,  $X, Y \in T_{\pi(\sigma)}M$  and  $V \in \mathcal{V}_\sigma$ . Then:*

$$\begin{aligned} (\nabla_{X^h} \Omega_{k,t})(Y^h, V)_\sigma &= \frac{t}{2} [(-1)^k g(\mathcal{R}(V), X \wedge Y) + g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y)] \\ (\nabla_V \Omega_{k,t})(X^h, Y^h)_\sigma &= -2g(V, X \wedge Y) - \frac{t}{2} g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y + K_\sigma X \wedge Y) \\ (\nabla_A \Omega_{k,t})(B, C) &= 0 \end{aligned}$$

when  $A, B, C$  are horizontal vectors or at least two of them are vertical.

**Proof:** Extend  $X, Y$  to vector fields in a neighborhood of the point  $p = \pi(\sigma)$ . Then, by (3.4), (3.5) and (3.2), we have

$$(\nabla_{X^h} \Omega_{k,t})(Y^h, V) = -h_t(\nabla_{X^h} Y^h, \mathcal{J}_k V) + h_t(\mathcal{J}_k Y^h, \nabla_{X^h} V)$$

$$\begin{aligned}
&= (-1)^k \frac{t}{2} g(R(X, Y)\sigma, \sigma \times V) + h_t((K_\sigma Y)^h, [X^h, V] + \nabla_V X^h) \\
&= (-1)^k \frac{t}{2} g(\mathcal{R}(V), X \wedge Y) + \frac{t}{2} g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y).
\end{aligned}$$

Next, by (3.5), we have

$$(\nabla_V \Omega_{k,t})(X^h, Y^h) = V h_t(X^h, \mathcal{J}_k Y^h) - \frac{t}{2} g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y + K_\sigma X \wedge Y)$$

Moreover,  $h_t(X^h, \mathcal{J}_k Y^h) = 2 \sum_{a=1}^3 y_a (g(s_a, Y \wedge X) \circ \pi)$ , hence  $V h_t(X^h, \mathcal{J}_k Y^h) = g(V, Y \wedge X)$ .

Let  $U, V, W$  be vertical vector fields on  $\mathcal{Z}$  near the point  $\sigma$ . Then

$$(\nabla_U \Omega_{k,t})(V, W) = 0$$

since the fibres of  $\mathcal{Z}$  are totally geodesic submanifolds and the restriction of  $\mathcal{J}_k$  on each fibre is Kählerian. We also have  $(\nabla_U \Omega_{k,t})(X^h, V) = 0$  in view of (3.5) and the fact that  $\nabla_U V$  is a vertical vector field. Next, by (3.5), we have:

$$\begin{aligned}
(\nabla_{X^h} \Omega_{k,t})(V, W) &= h_t(V, \nabla_{X^h} \mathcal{J}_k W - \mathcal{J}_k \nabla_{X^h} W) \\
&= h_t(V, [X^h, \mathcal{J}_k W] - \mathcal{J}_k [X^h, W]) = 0
\end{aligned}$$

since, as is easy to see,  $[X^h, \mathcal{J}_k W] = \mathcal{J}_k [X^h, W]$ . Indeed, take an oriented orthonormal frame  $\mathbf{e}_1, \dots, \mathbf{e}_4$  of  $TM$  near the point  $p = \pi(\sigma)$  such that  $D\mathbf{e}_i|_p = 0$ ,  $1 \leq i \leq 4$ , and  $s_1(p) = \sigma$  ( $s_1, s_2, s_3$  are defined by means of  $\mathbf{e}_1, \dots, \mathbf{e}_4$  as in the beginning of this section). The vector fields

$$U = y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2}, \quad J_1 U = y_1 y_3 \frac{\partial}{\partial y_1} + y_2 y_3 \frac{\partial}{\partial y_2} + (1 + y_3^2) \frac{\partial}{\partial y_3}$$

form a frame for the vertical bundle on  $\mathcal{Z}$  near  $\sigma$ . Since  $Ds_k|_p = 0$ , we have  $[X^h, U]_\sigma = [X^h, \mathcal{J}_1 U]_\sigma = 0$ . It follows that for every vertical vector field  $W$ ,  $[X^h, \mathcal{J}_k W]_\sigma = \mathcal{J}_k [X^h, W]_\sigma$ .

Finally, let  $X, Y, Z$  be vector fields on  $M$  near  $p$ . Then

$$(\nabla_{X^h} \Omega_{k,t})(Y^h, Z^h)_\sigma = -g(D_X s_1, Y \wedge Z)_p = 0$$

since  $D_X s_1|_p = 0$ . ■

Recall that

$$\begin{aligned}
h_t(N_k(A, B), C) &= -(\nabla_A \Omega_{k,t})(B, \mathcal{J}_k C) + (\nabla_B \Omega_{k,t})(A, \mathcal{J}_k C) \\
&\quad - (\nabla_{\mathcal{J}_k A} \Omega_{k,t})(B, C) + (\nabla_{\mathcal{J}_k B} \Omega_{k,t})(A, C)
\end{aligned}$$

where  $N_k$  is the Nijenhuis tensor of the almost complex structure  $\mathcal{J}_k$ . Then Lemma 2, (3.2) and (3.3) imply the following:

**Corollary 1** *Let  $\sigma \in \mathcal{Z}$ ,  $X, Y \in T_{\pi(\sigma)}M$  and  $V, W \in \mathcal{V}_\sigma$ . Then:*

$$\begin{aligned} N_k(X^h, Y^h)_\sigma &= R(X \wedge Y - K_\sigma X \wedge K_\sigma Y)_\sigma + (-1)^{k-1} \sigma \times R(K_\sigma X \wedge Y + X \wedge K_\sigma Y)_\sigma, \\ N_k(X^h, V)_\sigma &= 2[-1 + (-1)^{k-1}]g(V, X \wedge K_\sigma Y), \end{aligned}$$

$$N_k(V, W) = 0.$$

**Corollary 2** *Let  $A, B, C \in T_\sigma \mathcal{Z}$  and set  $X = \pi_* A, Y = \pi_* B, Z = \pi_* C$  and  $U = \mathcal{V}_\sigma A, V = \mathcal{V}_\sigma B, W = \mathcal{V}_\sigma C$ . Then:*

$$\begin{aligned} 3d\Omega_{k,t}(A, B, C) &= t(-1)^k [g(R(U), Y \wedge Z) + g(R(V), Z \wedge X) + g(R(W), X \wedge Y)] \\ &\quad - 2[g(U, Y \wedge Z) + g(V, Z \wedge X) + g(W, X \wedge Z)]. \end{aligned}$$

**Corollary 3** *Let  $A \in T_\sigma \mathcal{Z}$  and  $U = \mathcal{V}_\sigma A$ . Then the co-differential of  $\Omega_{k,t}$  is given by*

$$\delta\Omega_{k,t}(A) = tg(R(\sigma)\sigma, U).$$

**Theorem 2** *On the hyperbolic twistor space  $\mathcal{Z}$  of an oriented 4-dimensional manifold  $M$  with a neutral metric  $g$  we have the following:*

- (i) *The almost complex structure  $\mathcal{J}_1$  is integrable if and only if the metric  $g$  is self-dual. The almost Hermitian structure  $(\mathcal{J}_1, h_t)$  is semi-Kähler if and only if  $g$  is self-dual. It is indefinite Kähler if and only if the metric  $g$  is Einstein, self-dual, and  $t\tau = -12$ .*
- (ii) *The almost complex structure  $\mathcal{J}_2$  is never integrable. The almost Hermitian structure  $(\mathcal{J}_2, h_t)$  is semi-Kähler. It is indefinite almost Kähler or nearly Kähler if and only if the metric  $g$  is Einstein, self-dual and  $t\tau = 12$  or  $t\tau = -6$ , respectively.*

**Proof:** To see when  $\mathcal{J}_1$  is integrable, let us note first that the vertical space at any point  $\sigma \in \mathcal{Z}$  is spanned by the vectors of the form  $V = X \wedge Y - K_\sigma X \wedge K_\sigma Y, X, Y \in T_{\pi(\sigma)}M$ . Moreover, if  $V$  is of this form, then  $\sigma \times V = K_\sigma X \wedge Y + X \wedge K_\sigma Y$ . Therefore, by Corollary 1, the Nijenhuis tensor of  $\mathcal{J}_1$  vanishes if and only if

$$R(V)\sigma + \sigma \times R(\sigma \times V)\sigma = 0$$

for every  $\sigma \in \mathcal{Z}$  and  $V \in \mathcal{V}_\sigma$ . In view of (3.2), this is equivalent to

$$g(\mathcal{R}(V), W) = g(\mathcal{R}(\sigma \times V), \sigma \times W)$$

for every  $\sigma \in \mathcal{Z}, V, W \in \mathcal{V}_\sigma$ . Now, varying  $\sigma = y_1 s_1 + y_2 s_2 + y_3 s_3$  on the fibre  $y_1^2 - y_2^2 - y_3^2 = 1$  of  $\mathcal{Z}$  over a point  $p \in M$ , we see that the latter condition is satisfied if and only if  $g(\mathcal{R}(s_1), s_1) = -g(\mathcal{R}(s_2), s_2) = -g(\mathcal{R}(s_3), s_3)$  and  $g(\mathcal{R}(s_i), s_j) = 0$  for  $i \neq j$ . These identities are equivalent to the self-duality of the metric  $g$ .

The second identity of Corollary 1 shows that the almost complex structure  $\mathcal{J}_2$  is never integrable.

Corollary 3 and (3.2) imply that  $(\mathcal{J}_k, h_t)$ ,  $k = 1, 2$ , is semi-Kähler (i.e.  $\delta\Omega_{k,t} = 0$ ) if and only if  $g(\mathcal{W}^-(\sigma), \sigma \times U) = 0$  for every  $\sigma \in \mathcal{Z}$  and  $U \in \mathcal{V}_\sigma$  which is equivalent to  $\mathcal{W}^- = 0$ .

It follows from Corollary 2 that the fundamental 2-form of the almost Hermitian structure  $(\mathcal{J}_k, h_t)$  is closed if and only if for any  $\sigma \in \mathcal{Z}$  and  $V \in \mathcal{V}_\sigma$  we have

$$(-1)^k t \mathcal{R}(V) - 2V = 0$$

This is equivalent to  $g$  being Einstein, self-dual metric with  $t\tau = 12(-1)^k$ . In this case the structure  $(\mathcal{J}_1, h_t)$  is indefinite Kähler since the almost complex structure  $\mathcal{J}_1$  is integrable.

If the structure  $(\mathcal{J}_2, h_t)$  is nearly Kähler, then

$$(\nabla_{X^h} \Omega_{2,t})(Y^h, V)_\sigma + (\nabla_{Y^h} \Omega_{2,t})(X^h, V)_\sigma = 0$$

for every  $\sigma \in \mathcal{Z}$ ,  $V \in \mathcal{V}_\sigma$ ,  $X, Y \in T_{\pi(\sigma)}M$ . This identity and Lemma 2 imply

$$g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y - K_\sigma X \wedge Y) = 0$$

Now, taking into account Lemma 2, (3.3) and the latter equality, we obtain

$$\begin{aligned} 0 &= (\nabla_V \Omega_{2,t})(X^h, Y^h)_\sigma - (\nabla_{X^h} \Omega_{2,t})(Y^h, V)_\sigma = \\ &= -g(V, X \wedge Y - K_\sigma Y \wedge K_\sigma X) - \frac{t}{4} g(\mathcal{R}(V), X \wedge Y - K_\sigma Y \wedge K_\sigma X) \\ &\quad - \frac{3t}{4} g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma Y + K_\sigma X \wedge Y) \end{aligned} \quad (3.6)$$

As we have mentioned, the vectors of the form  $W = X \wedge Y - K_\sigma X \wedge K_\sigma Y$ ,  $X, Y \in T_{\pi(\sigma)}M$ , span the vertical space  $\mathcal{V}_\sigma$  and  $\sigma \times W = X \wedge K_\sigma Y + K_\sigma X \wedge Y$ . Therefore (3.6) is equivalent to

$$g(V, W) + \frac{t}{4} g(\mathcal{R}(V), W) + \frac{3t}{4} g(\mathcal{R}(\sigma \times V), \sigma \times W) = 0$$

for every  $V, W \in \mathcal{V}_\sigma$ . Varying  $\sigma$  on the fibres of  $\mathcal{Z}$  we see that  $g$  is Einstein and self-dual, and that  $t\tau = -6$ .

Conversely, it is not hard to show that under these conditions the structure  $(\mathcal{J}_k, h_t)$  on the twistor space  $\mathcal{Z}$  is nearly-Kähler.  $\blacksquare$

#### 4. Holomorphic functions

On the classical twistor space over a Riemannian 4-manifold with either almost complex structure, there are no global non-constant holomorphic functions, even when the base manifold is non-compact [6, 7]. However for the hyperbolic twistor space there is considerable difference from the classical case as we shall see.

First we remark that for local existence of holomorphic functions, the situation is the same in both the classical and hyperbolic cases. A  $C^\infty$  function on an almost complex manifold is said to be *holomorphic* if its differential is complex-linear with respect to the almost complex structure. On the twistor spaces, for any  $n = 0, 1, 2, 3$ , let  $\mathcal{F}_n(\mathcal{J}_i)$  denote the (possibly empty) set of points  $\sigma$  such that  $n$  is the maximal number of local  $\mathcal{J}_i$ -holomorphic functions with  $\mathbb{C}$ -linearly independent differentials at  $\sigma$ . In [6] it is shown that for the classical twistor space  $\mathcal{Z}$  of an oriented Riemannian 4-manifold  $M$  we have  $\mathcal{Z} = \mathcal{F}_0(\mathcal{J}_1) \cup \mathcal{F}_3(\mathcal{J}_1) = \mathcal{F}_0(\mathcal{J}_2) \cup \mathcal{F}_1(\mathcal{J}_2)$ ; moreover

$$\mathcal{F}_3(\mathcal{J}_1) = \pi^{-1}(\text{Int}\{p \in M : \mathcal{W}_p^- = 0\}),$$

$$\mathcal{F}_1(\mathcal{J}_2) = \pi^{-1}(\text{Int}\{p \in M : \mathcal{R}_p = \mathcal{W}_p^+\})$$

The same arguments give this result for the hyperbolic twistor space as well.

Let  $M$  be a pseudo-Riemannian four-manifold with metric  $g$  of signature  $(2, 2)$ . We shall say that  $M$  is *neutral almost hyperhermitian* if it admits a globally defined almost quaternionic structure of the second kind  $(J_1, J_2, J_3)$  compatible with the metric  $g$ . If the structure tensors  $J_1, J_2, J_3$  are integrable (i.e. their Nijenhuis tensors vanish) the manifold is called a *neutral hyperhermitian* surface; if, moreover,  $J_1, J_2, J_3$  are parallel with respect to the Levi-Civita connection of  $g$ ,  $M$  is called a *neutral hyperkähler* surface.

Suppose  $(M, g, J_1, J_2, J_3)$  is a neutral almost hyperhermitian four-manifold. Then all almost complex structures  $J_y = y_1 J_1 + y_2 J_2 + y_3 J_3$ ,  $y \in H = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 - y_2^2 - y_3^2 = 1\}$ , are compatible with the metric  $g$  and determine the same orientation on  $M$ . We shall always consider  $M$  with this orientation. As in the hyperhermitian case, if  $M$  is a neutral hyperhermitian surface, the metric  $g$  is self-dual and every almost complex structure  $J_y$ ,  $y \in H$ , is integrable. If, moreover,  $M$  is neutral hyperkähler, then it is indefinite Kähler and Ricci flat. It has been observed by Kamada [14] (see also [17]) that any compact neutral hyperkähler surface is biholomorphic to a complex torus or a primary Kodaira surface. He has also obtained a description of all neutral hyperkähler structures on the latter surfaces.

Given a neutral almost hyperhermitian four-manifold  $(M, g, J_1, J_2, J_3)$ , denote by  $\pi : \mathcal{Z} \rightarrow M$  the hyperbolic twistor space of  $(M, g)$ . The 2-vectors corresponding to  $J_1, J_2, J_3$  via (3.1) form a global frame of  $\bigwedge^-$  and we have a natural projection  $p : \mathcal{Z} \rightarrow H$  defined by  $p(\sigma) = (y_1, y_2, y_3)$  where  $K_\sigma = y_1 J_1(x) + y_2 J_2(x) + y_3 J_3(x)$ ,  $x = \pi(\sigma)$ . Thus  $\mathcal{Z}$  is diffeomorphic to  $M \times H$  by the map  $\sigma \rightarrow (\pi(\sigma), p(\sigma))$ . Further, we shall consider the hyperboloid  $H$  with the complex structure  $S$  determined by the restriction to  $H$  of the metric  $-dy_1^2 + dy_2^2 + dy_3^2$  of  $\mathbb{R}^3$ , i.e.  $SV = y \times V$  for  $V \in T_y H$  where  $\times$  is the vector product on  $\mathbb{R}^3$  defined by means of the paraquaternionic algebra. It is obvious that  $p$  maps any fibre of  $\mathcal{Z}$  biholomorphically on  $H$  with respect to  $\mathcal{J}_1$  and  $S$ .

The hyperboloid  $H$  has two connected components  $H^\pm = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 - y_2^2 - y_3^2 = 1, \pm y_1 > 0\}$  and the antipodal map  $y \rightarrow -y$  sends  $H^+$  anti-holomorphically onto  $H^-$ . Note also that the stereographic projec-

tion  $(y_1, y_2, y_3) \rightarrow \frac{y_2 + iy_3}{1 \pm y_1}$  of  $H^\pm$  from the point  $(\mp 1, 0, 0)$  is  $\pm$ -biholomorphic onto the unit disk  $\Delta$  in the complex plane  $\mathbb{C}$ .

**Theorem 3** *Let  $M$  be a neutral almost hyperhermitian four-manifold and  $\mathcal{Z}$  its hyperbolic twistor space. Then the natural projection  $p : \mathcal{Z} \rightarrow H$  is  $\mathcal{J}_1$ -holomorphic (resp.  $\mathcal{J}_2$ -anti-holomorphic) if and only if  $M$  is neutral hyperhermitian (resp. neutral hyperkähler).*

**Proof:** Let  $(g, J_1, J_2, J_3)$  be the neutral almost hyperhermitian structure on  $M$ .

As we have already mentioned, the restriction of  $p$  to any fibre of  $\mathcal{Z}$  is  $\mathcal{J}_1$ -holomorphic. Therefore  $p$  is  $\mathcal{J}_1$ -holomorphic on  $\mathcal{Z}$  if and only if  $p_*((K_\sigma X)_\sigma^h) = Sp_*(X_\sigma^h)$  for any  $\sigma \in \mathcal{Z}$  and  $X \in T_{\pi(\sigma)}M$ . Given  $\sigma \in \mathcal{Z}$ , the endomorphism  $K_\sigma$  has the form  $K_\sigma = y_1 J_1(x) + y_2 J_2(x) + y_3 J_3(x)$ ,  $x = \pi(\sigma)$ , where  $y = (y_1, y_2, y_3) \in H$  and we set  $J_y = y_1 J_1 + y_2 J_2 + y_3 J_3$ . Then, by (2.1), we have

$$p_*(X_\sigma^h) = - \sum_{b=1}^3 \epsilon_b (\langle D_X J_y, J_b \rangle \circ \pi) \frac{\partial}{\partial y_b}$$

and it follows that  $p$  is  $\mathcal{J}_1$  holomorphic if and only if

$$D_{J_y X} J_y = J_y D_X J_y$$

for any  $y \in H$  and  $X \in TM$ . The latter condition is equivalent to the almost complex structures  $J_y, y \in H$ , being integrable.

Similarly, the projection  $p$  is  $\mathcal{J}_2$ -anti-holomorphic if and only if

$$D_{J_y X} J_y = -J_y D_X J_y$$

for any  $y \in H$  and  $X \in TM$ . The latter condition is equivalent to the almost complex structures  $J_y$  being quasi Kähler. In dimension four this is equivalent to  $J_y$  being almost Kähler (i.e. with closed fundamental 2-forms).

Now the theorem follows from the following:

**Lemma 3**

- (i) *The almost complex structures  $J_y, y \in H$ , are integrable if and only if  $M$  is neutral hyperhermitian;*
- (ii) *The almost complex structures  $J_y, y \in H$ , are almost Kähler if and only if  $M$  is neutral hyperkähler.*

**Proof of the lemma:** The almost complex structure  $J_y$  is integrable if and only if  $D_{J_y X} J_y = J_y D_X J_y$  for any  $X \in TM$ . This identity is fulfilled for every  $y \in H$  if and only if

$$D_{J_k X} J_l + D_{J_l X} J_k = J_k D_X J_l + J_l D_X J_k, \quad 1 \leq k, l \leq 3.$$



Using (2.6) and the paraquaternionic identities for  $J_1, J_2, J_3$ , we see that, in the notation of (2.6), the latter equalities are satisfied if and only if  $\alpha(X) = -\beta(J_1X) = -\gamma(J_2X)$  which is equivalent to the integrability of the structure tensors  $J_1, J_2, J_3$ .

Proceeding in the same way, we see that  $D_{J_yX}J_y = -J_yD_XJ_y$  for any  $y \in H$  and  $X \in TM$  if and only if  $\alpha = \beta = \gamma = 0$ , i.e.  $DJ_1 = DJ_2 = DJ_3 = 0$ .  $\blacksquare$

**Corollary 4** *Let  $M$  be a compact neutral hyperhermitian manifold with hyperbolic twistor space  $\mathcal{Z}$  and let  $p : \mathcal{Z} \rightarrow H$  be the natural projection. Then any  $\mathcal{J}_1$ -holomorphic function  $f$  on  $\mathcal{Z}$  has the form  $f = g \circ p$  where  $g$  is a holomorphic function on  $H$ . If  $M$  is neutral hyperkähler, any  $\mathcal{J}_2$ -holomorphic function  $f$  on  $\mathcal{Z}$  has the form  $f = g \circ p$  where  $g$  is an anti-holomorphic function on  $H$ .*

**Proof:** Any global section  $s : N \rightarrow \mathcal{Z}$  of the hyperbolic twistor bundle of an oriented pseudo-Riemannian four-manifold  $N$  with a neutral metric determines a compatible almost complex structure  $K_s$  on  $N$  via (3.1) and vice versa. Since  $s_*(X) = X^h \circ s + D_X s$  for any  $X \in TN$ , it follows from (3.3) that the map  $s : (N, K_s) \rightarrow (\mathcal{Z}, \mathcal{J}_1)$  is holomorphic if and only if  $K_s$  is integrable;  $s : (N, K_s) \rightarrow (\mathcal{Z}, \mathcal{J}_2)$  is holomorphic if and only if  $K_s$  is almost Kähler.

Now let  $f$  be a  $\mathcal{J}_1$ -holomorphic function on the twistor space  $\mathcal{Z}$  of  $M$ . For any  $y \in H$ , denote by  $s_y$  the section of  $\mathcal{Z}$  determined by the almost complex structure  $J_y = y_1J_1 + y_2J_2 + y_3J_3$ . By Lemma 3(i), the structure  $J_y$  is integrable, hence the map  $s_y : (M, J_y) \rightarrow (\mathcal{Z}, \mathcal{J}_1)$  is holomorphic. Therefore  $f \circ s_y$  is a holomorphic function on the compact manifold  $M$ . So  $f \circ s_y$  is a constant and defining a function  $g$  on  $H$  by  $g(y) = f \circ s_y$  we have  $f = g \circ p$ . Since the restriction of  $p$  on a fibre of  $\mathcal{Z}$  is a biholomorphism onto  $H$ , the function  $g$  is holomorphic. Conversely, if  $g$  is a holomorphic function on  $H$ , then  $f = g \circ p$  is a  $\mathcal{J}_1$ -holomorphic function on  $\mathcal{Z}$  by Theorem 3.

Similar arguments prove the statement for the  $\mathcal{J}_2$ -holomorphic functions on  $\mathcal{Z}$ .  $\blacksquare$

In [17] J.Petean has classified the compact complex surfaces that admit indefinite Kähler-Einstein metrics. In particular, he has explicitly constructed Ricci flat (non-flat) examples of such metrics on complex tori, hyperelliptic surfaces and primary Kodaira surfaces.

Next we shall examine the J.Petean metrics on  $\mathbb{R}^4$ , a non-compact manifold. All of them have the form:

$$g = f(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) + dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2$$

where  $(x_1, x_2, x_3, x_4)$  are the standard coordinates on  $\mathbb{R}^4$  and  $f$  is a smooth positive function depending on  $x_1$  and  $x_2$  only. Consider the frame given by

$$\begin{aligned} \mathbf{e}_1 &= \frac{1}{\sqrt{f}} \frac{\partial}{\partial x_1}, & \mathbf{e}_2 &= \frac{1}{\sqrt{f}} \frac{\partial}{\partial x_2}, \\ \mathbf{e}_3 &= -\frac{1}{\sqrt{f}} \frac{\partial}{\partial x_1} + \sqrt{f} \frac{\partial}{\partial x_3}, & \mathbf{e}_4 &= -\frac{1}{\sqrt{f}} \frac{\partial}{\partial x_2} + \sqrt{f} \frac{\partial}{\partial x_4} \end{aligned}$$

Then  $\|\mathbf{e}_1\| = \|\mathbf{e}_2\| = -\|\mathbf{e}_3\| = -\|\mathbf{e}_4\| = 1$  and let  $s_1, s_2, s_3$  be the sections of  $\Lambda^-\mathbb{R}^4$  defined as in the last section. A direct computation shows that these sections are parallel, therefore they define a neutral hyperkähler structure on  $\mathbb{R}^4$ . So, the metric  $g$  is Ricci flat and self-dual [17], hence the almost complex structure  $\mathcal{J}_1$  on the hyperbolic twistor space  $\mathcal{Z}$  of  $(\mathbb{R}^4, g)$  is integrable. Moreover, we have the following result which shows that there can be an abundance of holomorphic functions on a hyperbolic twistor space.

**Theorem 4** *The hyperbolic twistor space  $(\mathcal{Z}, \mathcal{J}_1)$  of  $(\mathbb{R}^4, g)$  is biholomorphic to  $\mathbb{C}^2 \times H$ .*

**Proof:** Since the sections  $s_1, s_2, s_3$  of  $\mathcal{Z}$  are globally defined,  $\mathcal{Z}$  is diffeomorphic to  $\mathbb{R}^4 \times H$ . Denote by  $\mathcal{Z}^\pm$  the connected component of  $\mathcal{Z}$  determined by the hyperbolic plane  $H^\pm = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 - y_2^2 - y_3^2 = 1, \pm y_1 > 0\}$ . We shall identify  $H^\pm$  with the unit disk  $\Delta$  in the complex plane  $\mathbb{C}$  by means of the “stereographic” projection  $(y_1, y_2, y_3) \rightarrow \frac{y_2 \pm iy_3}{1 \pm y_1}$  and  $\mathcal{Z}^\pm$  with  $\mathbb{R}^4 \times \Delta$ . Our proof will be to show that there are global  $\mathcal{J}_1$ -holomorphic coordinates on  $\mathcal{Z}^\pm \cong \mathbb{R}^4 \times \Delta$ . We shall consider only the component  $\mathcal{Z}^+$  of  $\mathcal{Z}$  since the same reasoning works for  $\mathcal{Z}^-$ .

Since the sections  $s_1, s_2, s_3$  are parallel, it follows from (2.1) that the complex structure  $\mathcal{J}_1$  on  $\mathbb{R}^4 \times \Delta$  is given by

$$\begin{aligned} \mathcal{J}_1 \mathbf{e}_1 &= y_1 \mathbf{e}_2 + y_2 \mathbf{e}_3 + y_3 \mathbf{e}_4, & \mathcal{J}_1 \mathbf{e}_2 &= -y_1 \mathbf{e}_1 + y_3 \mathbf{e}_3 - y_2 \mathbf{e}_4, \\ \mathcal{J}_1 \mathbf{e}_3 &= y_2 \mathbf{e}_1 + y_3 \mathbf{e}_2 + y_1 \mathbf{e}_4, & \mathcal{J}_1 \mathbf{e}_4 &= y_3 \mathbf{e}_1 - y_2 \mathbf{e}_2 - y_1 \mathbf{e}_3, \\ \mathcal{J}_1 \frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, & \mathcal{J}_1 \frac{\partial}{\partial y} &= -\frac{\partial}{\partial x} \end{aligned}$$

where  $x, y$  are the standard coordinates on  $\Delta$  and  $y_1, y_2, y_3$  are defined by  $z = x + iy$  as follows

$$y_1 = \frac{1 + |z|^2}{1 - |z|^2}, \quad y_2 = \frac{z + \bar{z}}{1 - |z|^2}, \quad y_3 = \frac{z - \bar{z}}{i(1 - |z|^2)}.$$

Note that

$$\mathbf{e}_1 + i\mathcal{J}_1 \mathbf{e}_1, \quad \mathbf{e}_3 + i\mathcal{J}_1 \mathbf{e}_3, \quad \frac{\partial}{\partial x} + \mathcal{J}_1 \frac{\partial}{\partial x}$$

is a global frame of the bundle  $T^{0,1}(\mathbb{R}^4 \times \Delta)$  of  $(0, 1)$ -vectors with respect to  $\mathcal{J}_1$ . Thus a smooth complex-valued function  $G$  on  $\mathbb{R}^4 \times \Delta$  is  $\mathcal{J}_1$ -holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$(\mathbf{e}_1 + i\mathcal{J}_1 \mathbf{e}_1)G = (\mathbf{e}_3 + i\mathcal{J}_1 \mathbf{e}_3)G = \left(\frac{\partial}{\partial x} + \mathcal{J}_1 \frac{\partial}{\partial x}\right)G = 0.$$

Clearly the projection  $G_1 : \mathbb{R}^4 \times \Delta \rightarrow \Delta$  is a  $\mathcal{J}_1$ -holomorphic function, therefore any  $\mathcal{J}_1$ -holomorphic function  $G$  on  $\mathbb{R}^4 \times \Delta$  can be expanded in a series of  $z = x + iy$  with coefficients being smooth functions on  $\mathbb{R}^4$ . This remark leads us to seek a  $\mathcal{J}_1$ -holomorphic function which is linear in  $z$  and it is easy to

check that the function  $G_2 = (x_1 + ix_2) + iz(x_1 - ix_2)$  is  $\mathcal{J}_1$ -holomorphic. Next we shall show that there exists a third  $\mathcal{J}_1$ -holomorphic function  $G_3$  on  $\mathbb{R}^4 \times \Delta$  such that  $G_1, G_2, G_3$  form global  $\mathcal{J}_1$ -holomorphic coordinates on  $\mathbb{R}^4 \times \Delta$ . To do this, we take  $G_1$  and  $G_2$  as new coordinates, i.e. we introduce new smooth coordinates on  $\mathbb{R}^4 \times \Delta$  by setting

$$p = x_1(1 - y) + x_2x, \quad q = x_1x + x_2(1 + y), \quad r = x_3, \quad s = x_4, \quad u = x, \quad v = y \quad (4.1)$$

Set

$$F(p, q, u, v) = f\left(\frac{(1+v)p - uq}{1 - u^2 - v^2}, \frac{-up + (1-v)q}{1 - u^2 - v^2}\right) = f(x_1, x_2) \quad (4.2)$$

It is straightforward to compute the action of  $\mathcal{J}_1$  in the new coordinates (4.1) and to see that

$$\begin{aligned} \frac{\partial}{\partial p} + i\mathcal{J}_1 \frac{\partial}{\partial p} &= \frac{\partial}{\partial p} + i\frac{\partial}{\partial q} + i\frac{2uF}{(1 - u^2 - v^2)^2} \frac{\partial}{\partial r} + i\frac{2(u^2 + v^2 + v)}{(1 - u^2 - v^2)^2} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial r} + i\mathcal{J}_1 \frac{\partial}{\partial r} &= (1 + i\frac{2u}{1 - u^2 - v^2}) \frac{\partial}{\partial r} + i\frac{1 + u^2 + v^2 + 2v}{1 - u^2 - v^2} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial u} + i\mathcal{J}_1 \frac{\partial}{\partial u} &= \frac{\partial}{\partial u} + i\frac{\partial}{\partial v} + i\frac{2[(2u^2 + u^2v + v^3 - v)p - u(1 + u^2 + v^2 - 2v)q]F}{(1 - u^2 - v^2)^3} \frac{\partial}{\partial r} \\ &\quad + i\frac{2[u(1 + u^2 + v^2 + 2v)p - (2u^2 - u^2v - v^3 + v)q]F}{(1 - u^2 - v^2)^3} \frac{\partial}{\partial s} \end{aligned}$$

The vector fields

$$\frac{\partial}{\partial p} + i\mathcal{J}_1 \frac{\partial}{\partial p}, \quad \frac{\partial}{\partial r} + i\mathcal{J}_1 \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial u} + i\mathcal{J}_1 \frac{\partial}{\partial u}$$

form a global frame for the bundle  $T^{0,1}(\mathbb{R}^4 \times \Delta)$  of  $(0, 1)$ -vectors. Thus a smooth complex-valued function  $G(p, q, r, s, u, v)$  on  $\mathbb{R}^4 \times \Delta$  is  $\mathcal{J}_1$ -holomorphic if and only if the Cauchy-Riemann equations

$$(\frac{\partial}{\partial p} + i\mathcal{J}_1 \frac{\partial}{\partial p})G = (\frac{\partial}{\partial r} + i\mathcal{J}_1 \frac{\partial}{\partial r})G = (\frac{\partial}{\partial u} + i\mathcal{J}_1 \frac{\partial}{\partial u})G = 0$$

are satisfied. Now set  $z = u + iv$  and  $w = p + iq$ . Then a direct computation shows that  $G$  is  $\mathcal{J}_1$ -holomorphic if and only if

$$\frac{\partial G}{\partial s} = -i\frac{z - i}{z + i} \frac{\partial G}{\partial r} \quad (4.3)$$

$$\frac{\partial G}{\partial \bar{w}} = \frac{zF}{(1 - |z|^2)(z + i)} \frac{\partial G}{\partial r} \quad (4.4)$$

$$\frac{\partial G}{\partial \bar{z}} = \frac{(iw + z\bar{w})zF}{(1 - |z|^2)(z + i)} \frac{\partial G}{\partial r} \quad (4.5)$$

As we have mentioned the functions

$$G_1(p, q, r, s, u, v) = u + iv = z, \quad G_2(p, q, r, s, u, v) = p + iq = w$$

are  $\mathcal{J}_1$ -holomorphic. We shall seek a third holomorphic function  $G_3$  in the form

$$G_3(p, q, r, s, u, v) = r - i \frac{z-i}{z+i} s + H(p, q, u, v) \quad (4.6)$$

where  $H$  is a smooth function on  $\mathbb{R}^2 \times \Delta$ . The function  $G_3$  satisfies (4.3), (4.4) and (4.5) provided

$$\frac{\partial H}{\partial \bar{w}} = \frac{zF}{(1-|z|^2)(z+i)}, \quad \frac{\partial H}{\partial \bar{z}} = \frac{(iw + z\bar{w})zF}{(1-|z|^2)^2(z+i)} \quad (4.7)$$

This system (which is, in fact, a  $\bar{\partial}$ -equation on  $\mathbb{C} \times \Delta$ ) has a global solution if and only if

$$\frac{\partial}{\partial \bar{z}} \left( \frac{zF}{(1-|z|^2)(z+i)} \right) = \frac{\partial}{\partial \bar{w}} \left( \frac{(iw + z\bar{w})zF}{(1-|z|^2)^2(z+i)} \right)$$

which (in view of (4.2)) is equivalent to the following identity:

$$\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial \bar{z}} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial \bar{z}} = \frac{iw + z\bar{w}}{(1-|z|^2)} \left( \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial \bar{w}} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial \bar{w}} \right) \quad (4.8)$$

On the other hand

$$x_1 = \frac{(2i + z - \bar{z})(w + \bar{w}) - (z + \bar{z})(w - \bar{w})}{4i(1-|z|^2)}$$

and

$$x_2 = -\frac{(z + \bar{z})(w + \bar{w}) + (2i - z + \bar{z})(w - \bar{w})}{4(1-|z|^2)}.$$

Hence

$$\begin{aligned} \frac{\partial x_1}{\partial \bar{z}} &= \frac{(z+i)(w - iz\bar{w})}{2(1-|z|^2)^2}, & \frac{\partial x_2}{\partial \bar{z}} &= -i \frac{(z-i)(w - iz\bar{w})}{2(1-|z|^2)^2} \\ \frac{\partial x_1}{\partial \bar{w}} &= -i \frac{z+i}{2(1-|z|^2)}, & \frac{\partial x_2}{\partial \bar{w}} &= -\frac{z-i}{2(1-|z|^2)}. \end{aligned}$$

These identities show that

$$\frac{\partial x_1}{\partial \bar{z}} = \frac{iw + z\bar{w}}{1-|z|^2} \frac{\partial x_1}{\partial \bar{w}}, \quad \frac{\partial x_2}{\partial \bar{z}} = \frac{iw + z\bar{w}}{1-|z|^2} \frac{\partial x_2}{\partial \bar{w}}$$

and therefore identity (4.8) is satisfied.

Let  $H(p, q, u, v)$  be a global solution of (4.7). Consider the map  $(G_1, G_2, G_3) : (\mathbb{R}^4 \times \Delta, J_1) \rightarrow \mathbb{C}^2 \times \Delta$ . It is holomorphic and bijective since for any  $(\alpha, \beta, \gamma) \in \mathbb{C}^2 \times \Delta$  the system  $u + iv = \alpha, p + iq = \beta, r - i \frac{z-i}{z+i} s + H(p, q, u, v) = \gamma$  has a unique solution ( $u, v$  and  $p, q$  are uniquely determined by  $\alpha$  and  $\beta$ , then  $r, s$  are uniquely determined by  $\gamma - H(p, q, u, v)$  since  $i \frac{z-i}{z+i} \neq i \frac{z-i}{z+i}$  for every  $z$  with  $|z| \neq 1$ ).  $\blacksquare$

**Remark.** The choice of the function  $G_3$  is almost canonical since by the first equation of (4.7) it has the form

$$G_3 = H(u, v, p, q, r - i \frac{z-i}{z+i} s)$$

and it can be shown that  $H$  is a holomorphic function of  $t = r - i \frac{z-i}{z+i} s$ . Then for any fixed  $(u, v, p, q)$ ,  $t \rightarrow G_3(u, v, p, q, t)$  is a biholomorphism of  $\mathbb{C}$ , hence a linear function of  $t$ .

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